

Gradient expansion approach to nonlinear superhorizon perturbations II – a single scalar field –

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We formulate nonlinear perturbations of a scalar field dominated universe on super-horizon scales. We consider the case of a single scalar field. We take the gradient expansion approach. We adopt the uniform Hubble slicing and derive the general solution valid to $O(\epsilon^2)$, where ϵ is the expansion parameter associated with a spatial derivative, which includes both the scalar and tensor modes. In particular, the $O(\epsilon^2)$ correction terms to the nonlinear curvature perturbation, which become important in models with a non-slowroll stage during inflation, are explicitly obtained.

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I. INTRODUCTION

The cosmic microwave background (CMB) anisotropies recently observed by WMAP strongly support the inflationary cosmology, and theoretical predictions of various models of inflation seem to be well estimated by linear perturbation theory, with the cosmological perturbations generated from quantum vacuum fluctuations which are well approximated by Gaussian random fields [1]. Nevertheless, there has been a growing interest in detection of possible deviations from the Gaussian statistics. It was suggested that a deviation from the Gaussian statistics may be used to distinguish models of inflation [7], and it may indeed be detected by PLANCK [3] in the near future. Thus, making clear predictions on the non-Gaussianity from inflation have become one of the urgent issues of the inflationary cosmology. Since the nonlinearity is essential for the generation of non-Gaussian perturbations, it is necessary to develop a nonlinear cosmological perturbation theory to evaluate the non-Gaussianity from inflation.

Our goal is to formulate a nonlinear theory with which we can calculate non-Gaussianities from any models of inflation. The traditional approach is to develop a second-order perturbation theory [4, 5, 6, 8, 9]. However, here we adopt a different one, the gradient expansion approach, in which nonlinear perturbations are solved by invoking spatial gradient expansion under the assumption that spatial derivatives are small compared to time derivatives. Technically, we introduce an expansion parameter, ϵ , and associate it with each spatial derivative. Then we expand field equations in terms of ϵ and solve them order by order in ϵ iteratively. The gradient expansion approach has been developed and studied by many authors previously [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 29, 31].

In cosmological situations, the gradient expansion approach is valid on scales greater than the Hubble horizon scales, and an advantage of the approach is that we can calculate perturbations to full nonlinear order in their amplitudes. This is particularly useful when dealing with the general non-Gaussianity for which it may be necessary to evaluate not only second-order perturbative corrections but also higher order corrections.

At the leading order in the gradient expansion, i.e. neglecting all the spatial gradients, Lyth, Malik and Sasaki [11] studied nonlinear scalar curvature perturbations, proved the nonlinear δN formula, and constructed a gauge invariant (time-slice independent) nonlinear scalar curvature perturbation. Although this leading order approximation is sufficient for a large class of inflation models, there are models for which it is necessary to take into account the next order corrections. For example, in the context of the standard linear perturbation theory, Leach et al. pointed out that there can be enhancement of the comoving curvature perturbation on superhorizon scales, where it is usually conserved, even in a single field model if the slow-roll condition is temporarily violated [28]. There it was shown that $O(k^2)$ corrections to the curvature perturbation on superhorizon, where k is the comoving wavenumber, play a crucial role for the enhancement. Because $O(k^2)$ corrections correspond to $O(\epsilon^2)$ terms in gradient expansion, this implies that it is necessary to include $O(\epsilon^2)$ terms in such a model. Then, we expect that the enhancement may give rise to large non-Gaussianity. Indeed, Chen et al. [10] numerically found in a single-field inflation that large non-Gaussianity can be generated if the slow-roll condition is temporarily violated, using third-order action derived by Maldacena [5].

In this paper, focusing on a single-field inflation, we formulate the gradient expansion to $O(\epsilon^2)$ on the uniform Hubble slicing. In most of the previous studies, either only the leading order terms in gradient expansion was discussed or

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the choice of time-slicing was not quite adequate for the study of non-Gaussianity from inflation. Here we adopt the uniform Hubble slicing because the curvature perturbation on this slicing directly determines the initial condition for the CMB anisotropies and the large scale structure of the universe.

We employ the $(3+1)$ -decomposition of the Einstein equations, and consider a single scalar field with an arbitrary potential. We then derive the general solution for all the variables. As discussed in the case of a perfect fluid in [31], we find that the identification of the tensor mode in the spatial metric is rather arbitrary, depending on how one fixes the spatial coordinates, as a reflection of general covariance, while it can be unambiguously identified in the extrinsic curvature of the metric.

This paper is organized as follows. In Section II, we define basic variables, and describe the assumptions we adopt in gradient expansion. Then we present the general solution for all the physical quantities to $O(\epsilon^2)$ on the uniform Hubble slicing. In Section III, we discuss the validity of the assumption adopted in Section II by appealing to linear theory and by considering the vacuum fluctuations of the scalar field at and around horizon crossing. We conclude the paper in Section IV. In Appendix A, the basic equations are presented. In Appendix B, the estimation of the orders of physical quantities in powers of ϵ is given. In Appendix C, the general solutions for all the physical quantities are derived.

II. GRADIENT EXPANSION

A. Basic variables

In the $(3+1)$ -decomposition, the metric is expressed as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= (-\alpha^2 + \beta_k \beta^k) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j, \end{aligned} \quad (2.1)$$

where α , β^i ($\beta^i = \gamma^{ij} \beta_j$), and γ_{ij} are the lapse function, shift vector, and the spatial metric, respectively. We rewrite γ_{ij} as

$$\gamma_{ij}(t, x^k) = a^2(t) \psi^4(t, x^k) \tilde{\gamma}_{ij}(t, x^k); \quad \det(\tilde{\gamma}_{ij}) = 1, \quad (2.2)$$

where the function $a(t)$ is the scale factor of a fiducial homogeneous and isotropic background universe.

The extrinsic curvature K_{ij} is defined by

$$K_{ij} \equiv -\nabla_i n_j, \quad (2.3)$$

where $n_\mu = (-\alpha, 0, 0, 0)$ is the vector unit normal to the time slices. We decompose the extrinsic curvature as

$$K_{ij} = \frac{\gamma_{ij}}{3} K + \psi^4 a^2 \tilde{A}_{ij}; \quad K \equiv \gamma^{ij} K_{ij}, \quad (2.4)$$

where \tilde{A}_{ij} represents the traceless part of K_{ij} . The indices of K_{ij} are to be raised or lowered by γ^{ij} and γ_{ij} , and the indices of \tilde{A}_{ij} by $\tilde{\gamma}^{ij}$ and $\tilde{\gamma}_{ij}$.

The stress-energy tensor for a single scalar field is

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla^\alpha \phi \nabla_\alpha \phi + 2V(\phi)). \quad (2.5)$$

We define the local Hubble parameter as $1/3$ of the expansion of the unit normal vector n^μ , which is equal to $-1/3$ of the trace of the extrinsic curvature in our convention,

$$3H \equiv -K = \frac{3\dot{a}}{a} + 6 \frac{\partial_t \psi}{\alpha \psi} - \frac{D_i \beta^i}{\alpha}, \quad (2.6)$$

where a dot $\dot{}$ denotes d/dt and D_i is the covariant derivative with respect to γ_{ij} . In the following, we adopt the uniform Hubble slicing. For this slicing, we have

$$H(t) = \frac{\dot{a}}{a}, \quad (2.7)$$

and Eq. (2.6) implies

$$\alpha = 1 + \frac{2\partial_t \psi}{H\psi} - \frac{D_i \beta^i}{3H}. \quad (2.8)$$

B. Expansion scheme

We investigate nonlinear superhorizon perturbations with the gradient expansion approach, which is called by various names by various authors, the quasi-isotropic expansion [12, 13, 19, 21], the anti-Newtonian approximation [14, 15], the spatial gradient expansion [16, 17, 18, 20, 22, 23, 24, 25], or the long wavelength approximation [26, 29]. In this approach, we assume that the characteristic length scale L of inhomogeneities is always much larger than the Hubble horizon scale, $L \gg H^{-1} \sim t$. We introduce a small parameter ϵ , and assume that L is of $O(1/\epsilon)$. This assumption is equivalent to assuming that the magnitude of spatial gradients is given by $\partial_i \psi = \psi \times O(\epsilon)$, $\partial_i \alpha = \alpha \times O(\epsilon)$, etc.. In the limit $L \rightarrow \infty$, i.e., $\epsilon \rightarrow 0$, the universe looks locally like a FLRW spacetime, where 'locally' means as seen on the scale of the Hubble horizon volume. It is noted that physical quantities which are approximately homogeneous on each Hubble horizon scale can vary nonlinearly on very large scales.

The local homogeneity and isotropy imply that $\beta^i = O(\epsilon)$ and $\partial_t \tilde{\gamma}_{ij} = O(\epsilon)$ because the local FLRW equations should be realized in the limit $\epsilon \rightarrow 0$. However, we further assume that $\beta^i = O(\epsilon^3)$ and $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$. Technically, these additional assumptions make the analysis of $O(\epsilon^2)$ correction terms much simpler, as discussed in Appendix C.1. Physically, of course, it is necessary to justify them. For the former assumption on β^i , since it is just a matter of choice of the spatial coordinates, this does not cause any loss of generality. In fact, once we obtain the solution, it is straightforward to express it in a more general spatial coordinate system by the coordinate transformation $x^i \rightarrow \bar{x}^i = F^i(t, x^k)$ such that $\partial_t F^i = O(\epsilon)$, corresponding to $\bar{\beta}^i = O(\epsilon)$ in the new coordinate system. As for the latter assumption on $\partial_t \tilde{\gamma}_{ij}$, however, it is not simply a matter of choice. In Sec. III, with the help of the result from linear theory, we give a convincing argument, if not rigorous, that this assumption is indeed satisfied for perturbations arising from the vacuum fluctuations. To summarize, our basic assumptions are

$$\beta^i = O(\epsilon^3), \quad \partial_t \tilde{\gamma}_{ij} = O(\epsilon^2). \quad (2.9)$$

Applying these assumptions to the Einstein-scalar field equations, we find

$$\psi = O(1), \quad \alpha - 1 = O(\epsilon^2), \quad \partial_t \psi = O(\epsilon^2), \quad \tilde{A}_{ij} = O(\epsilon^2). \quad (2.10)$$

These estimates are derived in Appendix B. Here one comment is in order. The fact that $\partial_t \psi = O(\epsilon^2)$ means ψ is conserved if the $O(\epsilon^2)$ corrections can be neglected. This result was derived by Salopek and Bond [16] for a single scalar field system, and by Lyth, Malik, and Sasaki [11] for more general systems.

C. General solution

Here we present the general solution for all the physical quantities, valid to $O(\epsilon^2)$ in gradient expansion. We defer the derivation to Appendix C because it is not much different from the one we gave in the previous paper [31], except for the fact that the present paper deals with the case of a scalar field while the previous paper dealt with a perfect fluid.

The general solution is

$$\alpha = 1 + 2 \frac{\dot{\phi}_*}{a^3 \dot{\phi}^3} \left[{}_{(2)}C(x^k) \left(2a \dot{\phi} + \frac{dV}{d\phi} \int_{t_*}^t a(t') dt' \right) + {}_{(2)}D(x^k) \frac{dV}{d\phi} \right], \quad (2.11)$$

$$\psi = {}_{(0)}L(x^k) \left(1 + \frac{1}{2} \int_{t_*}^t (\alpha - 1) H dt' \right), \quad (2.12)$$

$$\tilde{\gamma}_{ij} = {}_{(0)}f_{ik}(x^\ell) \left(\delta_j^k - 2 {}_{(2)}F^k{}_j(x^\ell) \int_{t_*}^t \frac{dt'}{a^3(t')} \int_{t_*}^{t'} a(t'') dt'' - 2 {}_{(2)}C^k{}_j(x^\ell) \int_{t_*}^t \frac{dt'}{a^3(t')} \right), \quad (2.13)$$

$$\tilde{A}_{ij} = \frac{{}_{(2)}F_{ij}(x^k)}{a^3} \int_{t_*}^t a(t') dt' + \frac{{}_{(2)}C_{ij}(x^k)}{a^3}, \quad (2.14)$$

$$\tilde{\phi} = \phi(t) + {}_{(2)}C(x^k) \frac{\dot{\phi}_*}{a^3 \dot{\phi}^3} \int_{t_*}^t a(t') dt' + {}_{(2)}D(x^k) \frac{\dot{\phi}_*}{a^3 \dot{\phi}^3}, \quad (2.15)$$

where the index (n) is attached to a quantity of $O(\epsilon^n)$ except for the scalar field. For notational simplicity, the full scalar field is denoted by $\tilde{\phi}$, and the lowest order scalar field ${}_{(0)}\phi$, which depends only on time, is denoted simply by ϕ . The time t_* is an arbitrary reference time and $\dot{\phi}_* = \dot{\phi}(t_*)$. The tensor ${}_{(2)}F_{ij}$ and the scalar ${}_{(2)}C$ are given by Eqs. (C14) and (C34), respectively, as functions of ${}_{(0)}L$ and ${}_{(0)}f_{ij}$. The tensor ${}_{(2)}C_{ij}$ is traceless with respect to ${}_{(0)}f_{ij}$. The function ${}_{(2)}D$ is related to ${}_{(2)}C_{ij}$ through the momentum constraint (C36).

To clarify the physical role of these freely specifiable functions, let us first count the degrees of freedom. Since ${}_{(2)}C$ and ${}_{(2)}F_{ij}$ are determined by ${}_{(0)}L$ and ${}_{(0)}f_{ij}$, they have no degree of freedom. Since the determinant of ${}_{(0)}f_{ij}$ is unity, it has 5 degrees of freedom, and since ${}_{(2)}C_{ij}$ is traceless, it also has 5 degrees of freedom. In addition we have ${}_{(0)}L$ and ${}_{(2)}D$, each of which counts 1 degree of freedom. The momentum constraints which consist of 3 equations, relate ${}_{(2)}C_{ij}$ to ${}_{(2)}D$, and reduce the total degrees of freedom by 3. So, adding together, the total number is $5 + 5 + 1 + 1 - 3 = 9$, while the true physical degrees of freedom are $2 + 2 + 2 = 6$, where 2 are of the single scalar field (1 for a growing mode and 1 for a decaying mode) and $2 + 2$ are of the gravitational waves (2 for the metric and 2 for the extrinsic curvature). This implies that there still remains 3 degrees of freedom.

As discussed in the case of a single perfect fluid in the previous paper [31], the remaining 3 degrees of freedom come from the spatial covariance, that is, from the gauge freedom of purely spatial coordinate transformations $x^i \rightarrow \bar{x}^i = f^i(x^j)$. Thus we may regard that ${}_{(0)}f_{ij}$ contains these 3 gauge degrees of freedom. The correspondence between the nonlinear solutions and the solutions in linear theory can be understood from the time dependence of the nonlinear solutions. So, we see that ${}_{(0)}L$ and ${}_{(0)}f_{ij}$ represent growing modes, and ${}_{(2)}D$ and ${}_{(2)}C_{ij}$ represent decaying modes.

To summarize, the degrees of freedom contained in the freely specifiable functions can be interpreted as

$$\begin{aligned} {}_{(0)}L & \cdots 1 = 1 \text{ (scalar growing mode)}, \\ {}_{(0)}f_{ij} & \cdots 5 = 2 \text{ (tensor growing modes)} + 3 \text{ (gauge modes)}, \\ {}_{(2)}C_{ij} & \cdots 5 = 2 \text{ (tensor decaying modes)} + 3 \text{ (constraints)}, \\ {}_{(2)}D & \cdots 1 = 1 \text{ (scalar decaying mode)}. \end{aligned} \quad (2.16)$$

To fix the gauge completely, one has to impose 3 spatial gauge conditions on ${}_{(0)}f_{ij}$ to extract the physical tensor degrees of freedom. As discussed in Ref. [31], this cannot be done in a spatially covariant way because ${}_{(0)}f_{ij}$ is the metric and any covariant derivative of it vanishes identically. Thus the tensor modes cannot be extracted out from ${}_{(0)}f_{ij}$ unless we introduce a certain 'background' metric. This is an important difference from the linear case in which there exists a background metric.

In contrast, as for the extrinsic curvature, we may identify the tensor modes in it, because its transverse-traceless part can be extracted unambiguously [30, 31]. Namely, the transverse part of ${}_{(0)}L^\delta \tilde{A}_{ij}$ can be determined uniquely and it can be identified as the tensor modes. As noted in Ref. [31], this implies that the tensor modes in the extrinsic curvature are determined non-locally, and they exist even for a trivial ${}_{(0)}f_{ij}$, say for ${}_{(0)}f_{ij} = \delta_{ij}$. This generation of tensor modes in the extrinsic curvature is a result of nonlinear interactions of the scalar modes. We note, however, that it is not obvious if the tensor modes we identified can be called gravitational waves. To make a clear connection between the tensor modes and gravitational waves, it is necessary to evolve the system until the scale of interest becomes sufficiently smaller than the Hubble scale. But this is beyond the scope of the present paper.

III. VALIDITY OF THE ASSUMPTION IN THE LINEAR LIMIT

In this section, we discuss the validity of our central assumption, $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$. With the help of linear theory, we argue that it can be physically justified. We first consider the equation for the curvature perturbation during inflation, and argue that the condition $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$ is naturally satisfied. Then considering the quantum fluctuations as the source of the curvature perturbation, we explicitly show that this assumption holds not only in the case of slow-roll inflation but also in the case of the Starobinsky model, in which the slow-roll condition is temporarily violated due to a sudden change in the slope of the potential.

Since gradient expansion is effective only on superhorizon scales, as the initial condition for the quantities we calculate, we need to know their behavior when their wavelength crosses the Hubble horizon radius. To do so, we assume that the linear perturbation is a sufficiently good approximation up to and around the Hubble horizon scale. For linearized quantities, we follow the notation of Kodama and Sasaki [2].

As usual, we expand the linearized quantities in terms of spatial harmonics. The background is a spatially flat FLRW universe. We introduce the scalar, vector and tensor harmonics, $Y_{\mathbf{k}}$, $Y_{\mathbf{k}i}^{(1)}$ and $Y_{\mathbf{k}ij}^{(2)}$, respectively,

$$\begin{aligned} (\Delta + k^2)Y_{\mathbf{k}} &= 0, \\ (\Delta + k^2)Y_{\mathbf{k}i}^{(1)} &= 0; \quad \partial^i Y_{\mathbf{k}i}^{(1)} = 0, \\ (\Delta + k^2)Y_{\mathbf{k}ij}^{(2)} &= 0; \quad \delta^{ij}Y_{\mathbf{k}ij}^{(2)} = \partial^i Y_{\mathbf{k}ij}^{(2)} = 0, \end{aligned} \quad (3.1)$$

where $\Delta = \partial^i \partial_i$ is the 3-dimensional Laplacian. For the sake of notational simplicity, we suppress the mode index \mathbf{k} below. For a universe dominated by a scalar field, it is known that the vector perturbations are not excited. Therefore, we ignore the vector modes.

We express the metric as

$$ds^2 = a(\eta)^2 \left[-(1 + 2AY)d\eta^2 - 2BY_j dx^j d\eta + (\delta_{ij} + 2H_L Y \delta_{ij} + 2H_T Y_{ij} + 2H_T^{(2)} Y_{ij}^{(2)}) dx^i dx^j \right], \quad (3.2)$$

where $d\eta = dt/a(t)$ is the conformal time, and

$$Y_j \equiv -k^{-1} \partial_j Y, \quad Y_{ij} \equiv k^{-2} \left[\partial_i \partial_j - \frac{1}{3} \delta_{ij} \Delta \right] Y. \quad (3.3)$$

The correspondences of these to the metric components defined in Section II are

$$\alpha = 1 + AY, \quad \beta_j = -aBY_j, \quad \psi^2 = 1 + H_L Y, \quad \tilde{\gamma}_{ij} = \delta_{ij} + 2H_T Y_{ij} + 2H_T^{(2)} Y_{ij}^{(2)}. \quad (3.4)$$

In passing, we note that our assumption $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$ corresponds to $\partial_t H_T = O(\epsilon^2)$ and $\partial_t H_T^{(2)} = O(\epsilon^2)$ in the linear limit. We also introduce the quantities

$$\mathcal{R} \equiv H_L + \frac{H_T}{3}, \quad (3.5)$$

$$\sigma_g \equiv k^{-1} H'_T - B, \quad (3.6)$$

$$\mathcal{K} \equiv -A + \frac{k}{3\mathcal{H}} B + \mathcal{H}^{-1} H'_L = -A + \mathcal{H}^{-1} \mathcal{R}' - \frac{k}{3\mathcal{H}} \sigma_g, \quad (3.7)$$

where a prime (') denotes a conformal time derivative $d/d\eta$. These quantities are known to be independent of the choice of the spatial coordinates but depend only on the choice of time-slicing. The first one, \mathcal{R} is called the curvature perturbation because the spatial curvature scalar $R[\gamma]$ is given by

$$R[\gamma] = -\frac{4}{a^2} \Delta [\mathcal{R}Y]. \quad (3.8)$$

The variables \mathcal{K} and σ_g represent the perturbations in the extrinsic curvature,

$$K = -3H(1 + \mathcal{K}Y), \quad \tilde{A}_{ij} = -\frac{k}{a} \sigma_g Y_{ij}. \quad (3.9)$$

Under the gauge transformation induced by an infinitesimal change of time-slicing $\bar{\eta} = \eta + TY$, \mathcal{R} , σ_g and \mathcal{K} transform as

$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{H}T, \quad (3.10)$$

$$\bar{\sigma}_g = \sigma_g - kT, \quad (3.11)$$

$$\bar{\mathcal{K}} = \mathcal{K} + \mathcal{H}^{-1} \left(\mathcal{H}^2 - \mathcal{H}' + \frac{k^2}{3} \right) T, \quad (3.12)$$

where $\mathcal{H} \equiv aH$ is the conformal Hubble parameter.

Since the standard linear calculation gives the curvature perturbation on the comoving slicing, we need to relate the quantities on the comoving slicing to those on the uniform Hubble slicing. In particular, analytic expressions for the comoving curvature perturbations in the Starobinsky model resulting from the quantum fluctuations are given in [27]. Here and in what follows, we denote a quantity on the comoving and uniform Hubble slicing by the subscripts c and H , respectively. Thus, we first express the geometrical quantities on the comoving slicing in terms of \mathcal{R}_c . Then we consider an infinitesimal transformation from the comoving slicing to the uniform Hubble slicing.

The $(0, \mu)$ -components of the Einstein equations on the comoving slicing give

$$\begin{aligned} \delta G_0^0 &= \kappa^2 \delta T_0^0; \\ 2[3\mathcal{H}^2 A_c + \mathcal{H}k(\sigma_g)_c - 3\mathcal{H}\mathcal{R}'_c - k^2 \mathcal{R}_c] &= \kappa^2 \phi'^2 A_c, \end{aligned} \quad (3.13)$$

$$\begin{aligned} \delta G_j^0 &= \kappa^2 \delta T_j^0; \\ \mathcal{H}A_c - \mathcal{R}'_c &= 0, \end{aligned} \quad (3.14)$$

where $\kappa^2 = 8\pi G$. From these, we have

$$A_c = \frac{1}{\mathcal{H}} \mathcal{R}'_c, \quad k(\sigma_g)_c = \frac{1}{\mathcal{H}} \left(k^2 \mathcal{R}_c + \frac{\kappa^2}{2} \frac{\phi'^2}{\mathcal{H}} \mathcal{R}'_c \right), \quad (3.15)$$

where it is noted that the background equation gives the relation,

$$-a^2 \dot{H} = -a \left(\frac{\mathcal{H}}{a} \right)' = \mathcal{H}^2 - \mathcal{H}' = \frac{\kappa^2}{2} \phi'^2. \quad (3.16)$$

Then the second equality of Eq. (3.7) gives

$$\mathcal{K}_c = -\frac{k}{3\mathcal{H}}(\sigma_g)_c = -\frac{1}{3\mathcal{H}^2} \left(k^2 \mathcal{R}_c + \frac{\kappa^2 \phi'^2}{2\mathcal{H}} \mathcal{R}'_c \right). \quad (3.17)$$

Since \mathcal{K} is zero on the uniform Hubble slicing by definition, we have

$$0 = \mathcal{K}_H = \mathcal{K}_c + \mathcal{H}^{-1}(\mathcal{H}^2 - \mathcal{H}' + \frac{k^2}{3})T. \quad (3.18)$$

This gives T in terms of \mathcal{K}_c , which in turn is given by Eq. (3.17). The result is

$$\mathcal{H}T = \frac{1}{3\frac{\kappa^2}{2}\phi'^2 + k^2} \left(k^2 \mathcal{R}_c + \frac{\kappa^2 \phi'^2}{2\mathcal{H}} \mathcal{R}'_c \right). \quad (3.19)$$

Thus, we obtain

$$\mathcal{R}_H = \left(H_L - \frac{H_T}{3} \right)_H = \mathcal{R}_c - \frac{1}{3\frac{\kappa^2}{2}\phi'^2 + k^2} \left(k^2 \mathcal{R}_c + \frac{\kappa^2 \phi'^2}{2\mathcal{H}} \mathcal{R}'_c \right), \quad (3.20)$$

$$(k\sigma_g)_H = (H'_T - kB)_H = \frac{1}{\mathcal{H}} \frac{\frac{\kappa^2}{2}\phi'^2}{\frac{\kappa^2}{2}\phi'^2 + \frac{k^2}{3}} \left(k^2 \mathcal{R}_c + \frac{\kappa^2 \phi'^2}{2\mathcal{H}} \mathcal{R}'_c \right). \quad (3.21)$$

Since $B = O(k)$ because $\beta^i = O(\epsilon)$, the above results imply that our assumption $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$, which corresponds to $H'_T = O(k^2)H_T$ in the linear limit, is justified if $\mathcal{R}'_c = O(k^2)\mathcal{R}_c$ on superhorizon scales.

Now let us argue that this is indeed so in general. We know that \mathcal{R}_c satisfies the equation,

$$\mathcal{R}''_c + 2\frac{z'}{z}\mathcal{R}'_c + k^2\mathcal{R}_c = 0; \quad z \equiv \frac{a\phi'}{\mathcal{H}} = \frac{a\dot{\phi}}{H}. \quad (3.22)$$

We consider an inflationary universe in which we approximately have $\mathcal{H} \sim -1/\eta$. There are two independent solutions u and v , which are functions of $k\eta$. We may fix these solutions such that they behave in the superhorizon limit $k/\mathcal{H} \sim k|\eta| \rightarrow 0$ as

$$\begin{aligned} u &= [1 + O(k^2)] u_0; \quad u_0 = \text{const.}, \\ v &= [1 + O(k^2)] v_0; \quad v_0 \equiv \frac{\int_{\eta_k}^0 d\eta' / z^2(\eta')}{\int_{\eta_k}^0 d\eta' / z^2(\eta')}, \end{aligned} \quad (3.23)$$

where $\eta_k \sim -k^{-1}$ is the horizon crossing time, $\mathcal{H}(\eta_k) \equiv k$, and we have used the fact that the long wavelength expansion of Eq. (3.22) would give corrections only in powers of k^2 . Then the general solution for \mathcal{R}_c is given by

$$\mathcal{R}_c = Au + Bv, \quad (3.24)$$

where A and B are constants of order unity in the sense of the k expansion. Thus at $k|\eta| \ll 1$, we have

$$\mathcal{R}_c = Au_0, \quad \mathcal{R}'_c = O(k^2)Au_0 + Bv'_0. \quad (3.25)$$

Now, assuming that the violation of the slow-roll condition is not too strong, which is necessary for the spectrum to be approximately scale-invariant, we have $z \propto a \propto \eta^{-1}$, hence

$$v'_0 = - \left(\int_{\eta_k}^0 d\eta' \frac{z^2(\eta')}{z^2(\eta')} \right)^{-1} = O(k^3). \quad (3.26)$$

Thus we conclude that $\mathcal{R}'_c = O(k^2)\mathcal{R}_c$ on superhorizon scales.

To reinforce the above argument, let us consider the Starobinsky model in which the slow-roll condition can be significantly violated [27]. The Starobinsky model has a sudden change in its slope at $\phi = \phi_0$ such that

$$V(\phi) = \begin{cases} V_0 + A_+(\phi - \phi_0) & \text{for } \phi > \phi_0, \\ V_0 + A_-(\phi - \phi_0) & \text{for } \phi < \phi_0, \end{cases} \quad (3.27)$$

where A_+ , A_- and ϕ_0 are assumed to be positive so that the scalar field evolves from a large positive value of ϕ toward $\phi = 0$. Then the background scalar field ϕ satisfies

$$3H\dot{\phi} = \begin{cases} -A_+ & \text{for } \phi > \phi_0, \\ -(A_- + (A_+ - A_-)e^{-3H(t-t_0)}) & \text{for } \phi < \phi_0, \end{cases} \quad (3.28)$$

where t_0 is the time at which $\phi = \phi_0$, and the de Sitter approximation, $3H^2 = \kappa^2 V_0$ is assumed to be valid. Thus the scalar field slow-rolls at $\phi > \phi_0$, and violates the slow-roll condition temporarily at $\phi < \phi_0$. The evolution is decelerated if $A_+/A_- > 1$, or accelerated if $A_+/A_- < 1$, compared to the slow-roll evolution.

The curvature perturbation on the comoving slicing during the non-slow-roll regime at $\phi \leq \phi_0$ is [27]

$$\mathcal{R}_c = -\frac{iH^2}{\sqrt{2}k^{3/2}\dot{\phi}} \left(\alpha(k)e^{-ik\eta}(1 + ik\eta) - \beta(k)e^{ik\eta}(1 - ik\eta) \right), \quad (3.29)$$

$$\alpha(k) \equiv 1 + \frac{3i}{2} \left(\frac{A_-}{A_+} - 1 \right) \frac{k_0}{k} \left(1 + \frac{k_0^2}{k^2} \right), \quad (3.30)$$

$$\beta(k) \equiv -\frac{3i}{2} \left(\frac{A_-}{A_+} - 1 \right) e^{2ik/k_0} \frac{k_0}{k} \left(1 + \frac{ik_0}{k} \right)^2, \quad (3.31)$$

$$|\alpha|^2 - |\beta|^2 = 1,$$

where $k_0 = (a(t_0)H_0)^{-1}$. On superhorizon scales, we have $k|\eta| \ll 1$. The standard slow-roll case corresponds to $A_+ = A_-$. In this case, we immediately see that $\alpha(k) = 1$, $\beta(k) = 0$, and $\mathcal{R}'_c = O(k^2)\mathcal{R}_c$ on superhorizon scales.

We now turn to the case $A_+ \neq A_-$. We assume $A_-/A_+ = O(1)$ so that the slow-roll condition is not severely violated. For $k < k_0$, we expand the three exponentials in Eq. (3.29) to obtain

$$\mathcal{R}_c = \frac{3iH^3}{\sqrt{2}k^{3/2}A_+} \left(1 - \frac{A_+}{3H\dot{\phi}} \frac{(k\eta)^2}{2} \left[1 + \left(\frac{A_-}{A_+} - 1 \right) \left(1 + 2k_0\eta - \frac{(k_0\eta)^3}{5} - \frac{4}{5(k_0\eta)^2} \right) + O(k) \right] \right). \quad (3.32)$$

Thus we have $\mathcal{R}'_c = O(k^2)\mathcal{R}_c$. For $k > k_0$, we have $\alpha(k) = 1 + O(k_0/k)$ and $\beta(k) = O(k_0/k)$. Hence

$$\mathcal{R}'_c = \left[-a\frac{\ddot{\phi}}{\dot{\phi}} + O(k^2) \right] \mathcal{R}_c = \left[O((k\eta)^2) \left(\frac{k_0}{k} \right)^2 + O(k^2) \right] \mathcal{R}_c. \quad (3.33)$$

Since $k > k_0$, we see the right-hand side of this is also of $O(k^2)\mathcal{R}_c$. Thus we have shown that $\mathcal{R}'_c = O(k^2)\mathcal{R}_c$ holds even for the Starobinsky model in which the slow-roll condition can be violated. This in turn implies $H'_T = O(k^2)H_T$.

Next, we consider the tensor perturbation $H_T^{(2)}$. The evolution equation is

$$H_T^{(2)''} + 2\frac{a'}{a}H_T^{(2)'} + k^2H_T^{(2)} = 0. \quad (3.34)$$

This equation has the same structure as Eq. (3.22) for \mathcal{R}_c if we replace z with a . Then we can repeat exactly the same argument and conclude that $H_T^{(2)'} = O(k^2)H_T^{(2)}$.

Again, this conclusion can be supported by considering the quantum fluctuations explicitly. When the de Sitter approximation $H = \text{const.}$ holds, the tensor perturbation from the quantum fluctuations during inflation are given by

$$H_T^{(2)} = \frac{\sqrt{2\pi}H}{k^{3/2}}(k\eta - i)e^{-ik\eta}. \quad (3.35)$$

Taking the conformal time derivative of this equation, we obtain at $k|\eta| \ll 1$,

$$H_T^{(2)'} = -\frac{\sqrt{2\pi}H}{k^{3/2}}(ik^2\eta)e^{-ik\eta} = O(k^2)H_T^{(2)}. \quad (3.36)$$

Thus, provided that the linear theory is a good approximation up to scales corresponding to a few e-foldings after horizon crossing, our assumption $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$ is justified for perturbations originated from the quantum fluctuations inside the horizon.

IV. CONCLUSION

In this paper, taking the gradient expansion approach, we have investigated nonlinear perturbations on superhorizon scales in a universe dominated by a single scalar field. We have derived the general solution for all the physical quantities valid to second order in the spatial gradients on the uniform Hubble slicing. In particular, an expression for the nonlinear curvature perturbation, which plays a central role in the evaluation of a possible non-Gaussianity from inflation, has been obtained.

Parallel to our previous paper [31], we have identified the tensor modes in the extrinsic curvature, while the identification of the tensor modes in the metric is arbitrary unless we specify the background spatial metric. This is an important difference from the case of linear theory in which the background metric is uniquely given.

In our analysis, we have adopted a non-trivial assumption $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$, which cannot be justified within the context of gradient expansion. To justify it, we have appealed to linear theory, and argued that the assumption $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$ is naturally satisfied in the linear limit. In particular, as an explicit example, we have considered the Starobinsky model, in which the slow-roll condition is temporarily violated due to a sudden change in the slope of the potential, and we have explicitly shown that the quantum fluctuations indeed satisfy the assumption $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$.

As mentioned in Introduction, in models in which the slow-roll condition is temporarily violated as in the case of the Starobinsky model, the $O(k^2)$ corrections to the curvature perturbation, which correspond to $O(\epsilon^2)$ corrections in gradient expansion, may play a crucial role in the determination of the final amplitude of the curvature perturbation. Since the result of this paper is valid to $O(\epsilon^2)$, it provides a very useful tool to investigate the nonlinear behavior of the curvature perturbation on superhorizon scales for a wide class of models including those which violate the slow-roll condition. In particular, matching our result with the quantum fluctuations on scales slightly beyond the horizon scale, the non-Gaussianity arising from the nonlinear dynamics of the scalar field on superhorizon scales can be studied. Applications to specific models of inflation will be considered in a future publication.

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APPENDIX A: BASIC EQUATIONS

Here we show the basic equations for nonlinear quantities. The Klein-Gordon equation is

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left[\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \phi \right] - \frac{dV}{d\phi} = 0 \quad (\text{A1})$$

We employ the $(3+1)$ -formalism of the Einstein equations and then the dynamical variables are γ_{ij} and K_{ij} . The (n, n) and (n, j) components of the Einstein equations give the Hamiltonian and momentum constraint equations, respectively, while the (i, j) components gives the evolution equations for K_{ij} . The evolution equations for γ_{ij} are given by the definitions of the extrinsic curvature (2.3).

In the present case, the Hamiltonian and momentum constraints are

$$R - \tilde{A}_{ij} \tilde{A}^{ij} + \frac{2}{3} K^2 = 16\pi G E, \quad (\text{A2})$$

$$D_i \tilde{A}^i_j - \frac{2}{3} D_j K = 8\pi G J_j, \quad (\text{A3})$$

$$E \equiv T_{\mu\nu} n^\mu n^\nu, \quad J_j \equiv -T_{\mu\nu} n^\mu \gamma_j^\nu. \quad (\text{A4})$$

The evolution equations for γ_{ij} are given as

$$(\partial_t - \beta^k \partial_k) \psi + \frac{\dot{a}}{2a} \psi = \frac{\psi}{6} \{-\alpha K + \partial_k \beta^k\}, \quad (\text{A5})$$

$$(\partial_t - \beta^k \partial_k) \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik} \partial_j \beta^k + \tilde{\gamma}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{\gamma}_{ij} \partial_k \beta^k, \quad (\text{A6})$$

where $\dot{} = d/dt$.

The evolution equations for K_{ij} are given as

$$(\partial_t - \beta^k \partial_k)K = \alpha(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2) - D_k D^k \alpha + 4\pi G\alpha(E + S^k_k), \quad (\text{A7})$$

$$\begin{aligned} (\partial_t - \beta^k \partial_k)\tilde{A}_{ij} &= \frac{1}{a^2\psi^4}[\alpha(R_{ij} - \frac{\gamma_{ij}}{3}R) - (D_i D_j \alpha - \frac{\gamma_{ij}}{3}D_k D^k \alpha)] \\ &\quad + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j) \\ &\quad + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{jk}\partial_i \beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k \beta^k - \frac{8\pi G\alpha}{a^2\psi^4}(S_{ij} - \frac{\gamma_{ij}}{3}S^k_k), \end{aligned} \quad (\text{A8})$$

where R_{ij} is the Ricci tensor of the metric γ_{ij} , $R \equiv \gamma^{ij}R_{ij}$, D_i is the covariant derivative with respect to γ_{ij} , and

$$S_{ij} \equiv T_{ij}, \quad S^k_k \equiv \gamma^{kl}S_{lk}. \quad (\text{A9})$$

APPENDIX B: ORDER ESTIMATION

Here we evaluate the order of magnitude of the basic variables using the equations presented in Appendix A and the assumptions given by Eq. (2.9), namely

$$\beta^i = O(\epsilon^3), \quad \partial_t \tilde{\gamma}_{ij} = O(\epsilon^2). \quad (\text{B1})$$

With these assumptions, Eqs. (A3) and (A6) on the uniform Hubble slicing yield

$$J_j = O(\epsilon^3). \quad (\text{B2})$$

Using Eq. (2.5), J_j is express as

$$J_j = -\frac{1}{\alpha}\partial_0\phi\partial_j\phi + \frac{\beta^l}{\alpha}\partial_l\phi\partial_j\phi. \quad (\text{B3})$$

We expand ϕ as $\phi = {}_{(0)}\phi + {}_{(1)}\phi + {}_{(2)}\phi + \dots$. Then, to satisfy Eq.(B2), ${}_{(0)}\phi$ should depend only on time, and ${}_{(1)}\phi$ should vanish. Thus, we obtain

$${}_{(0)}\phi = {}_{(0)}\phi(t), \quad {}_{(1)}\phi = 0. \quad (\text{B4})$$

For notational simplicity, we denote the full scalar field by $\tilde{\phi}$ and ${}_{(0)}\phi$ by ϕ in the following. Thus

$$\tilde{\phi}(t, x^i) = \phi(t) + {}_{(2)}\phi(t, x^i) + \dots \quad (\text{B5})$$

From the $O(\epsilon^0)$ part of the Hamiltonian constraint, Eq. (A2), we have

$$\frac{1}{3}K^2(t) = 3H^2 = 8\pi G\left(\frac{\dot{\phi}^2}{2{}_{(0)}\alpha^2} + V(\phi)\right), \quad (\text{B6})$$

where we have also expanded α as ${}_{(0)}\alpha + {}_{(2)}\alpha + \dots$. From this equation, we find that ${}_{(0)}\alpha$ depend only on time,

$${}_{(0)}\alpha = {}_{(0)}\alpha(t). \quad (\text{B7})$$

Since ${}_{(0)}\alpha$ is spatial homogeneous, we may choose the time coordinate to set it to unity, ${}_{(0)}\alpha = 1$. Thus, from Eqs. (A5) and $3H = -K$ we have

$$0 = -3\frac{\dot{a}}{a}{}_{(2)}\alpha + \frac{6\dot{\psi}}{\psi}(1 - {}_{(2)}\alpha) + O(\epsilon^4). \quad (\text{B8})$$

We see $\partial_t \psi = O(\epsilon^2)$ from this equation.

To summarize, the orders of magnitude of the basic metric quantities are

$$\psi = O(1), \quad \beta^i = O(\epsilon), \quad \alpha - 1 = O(\epsilon^2), \quad \partial_t \tilde{\gamma}_{ij} = O(\epsilon^2), \quad \partial_t \psi = O(\epsilon^2), \quad \tilde{A}_{ij} = O(\epsilon^2). \quad (\text{B9})$$

Actually, as for β^i , when we solve the equations to $O(\epsilon^2)$ in Appendix C, we assume $\beta^i = O(\epsilon^3)$. But since the choice of β^i does not affect the temporal behavior, it does not affect the generality of the solution.

APPENDIX C: DERIVATION OF THE GENERAL SOLUTION

1. The leading order solution

We first derive the leading order solution for our basic variables. Here, the "leading order" means not the lowest order in gradient expansion, but the lowest order of each physical quantity. For example, from Eqs. (2.10), the leading order of ψ is $O(\epsilon^0)$, but the leading one of \tilde{A}_{ij} is $O(\epsilon^2)$.

The $O(\epsilon^0)$ part of Eqs. (A2) and (A7) are

$$\frac{1}{3}K^2(t) = 3H^2 = 8\pi G \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right), \quad (C1)$$

$$\dot{K} = -3\dot{H} = 3H^2 + 4\pi G \left(\dot{\phi}^2 - 2V(\phi) \right). \quad (C2)$$

These equations are indeed the Friedmann equations, and the leading solution ϕ is that of a FLRW spacetime.

Setting $\beta^i = O(\epsilon^3)$, we substitute the order of magnitude evaluation of the variables shown in Eq. (2.10) into the Klein-Gordon equation and find

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \quad (C3)$$

$${}_{(2)}\ddot{\phi} + 3H{}_{(2)}\dot{\phi} + \frac{d^2V}{d^2\phi}{}_{(2)}\phi - \dot{\phi}({}_{(2)}\dot{\alpha} + 3H{}_{(2)}\alpha) - 2\ddot{\phi}{}_{(2)}\alpha = 0. \quad (C4)$$

The Hamiltonian and momentum constraint equations give

$$\tilde{\gamma}^{ij}\tilde{D}_i\tilde{D}_j\psi = \frac{1}{8}\tilde{\gamma}^{kl}\tilde{R}_{kl}\psi - 2\pi G\psi^5a^2 \left[-\dot{\phi}^2{}_{(2)}\alpha + \dot{\phi}{}_{(2)}\dot{\phi} + \frac{dV}{d\phi}{}_{(2)}\phi \right] + O(\epsilon^4), \quad (C5)$$

$$\tilde{D}^j(\psi^6\tilde{A}_{ij}) = -8\pi G\psi^6\dot{\phi}\partial_j({}_{(2)}\phi) + O(\epsilon^5), \quad (C6)$$

where \tilde{R}_{ij} is the Ricci tensor with respect to $\tilde{\gamma}_{ij}$, and \tilde{D}_i is the covariant derivative with respect to $\tilde{\gamma}_{ij}$.

The evolution equations for the spatial metric give

$$\frac{6\partial_t\psi}{\psi} - 3H(\alpha - 1) = D_k\beta^k, \quad (C7)$$

$$(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} = -2\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k + O(\epsilon^4), \quad (C8)$$

while the evolution equations for the extrinsic curvature give

$$\partial_t\tilde{A}_{ij} + 3H\tilde{A}_{ij} = \frac{1}{a^2\psi^4}[R_{ij} - \frac{\gamma_{ij}}{3}R] + O(\epsilon^4), \quad (C9)$$

$$0 = 3H^2{}_{(2)}\alpha + 8\pi G \left[-\left(\dot{\phi}^2 + V(\phi)\right)_{(2)}\alpha + 2\dot{\phi}{}_{(2)}\dot{\phi} - \frac{dV}{d\phi}{}_{(2)}\phi \right]. \quad (C10)$$

First, for ψ , we have

$$\psi = {}_{(0)}L(x^i) + O(\epsilon^2), \quad (C11)$$

where ${}_{(0)}L$ is an arbitrary function of the spatial coordinates. And from Eqs. (C9) and (C11) together with the assumption $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$, we obtain

$$\tilde{\gamma}_{ij} = {}_{(0)}f_{ij}(x^k) + O(\epsilon^2), \quad (C12)$$

$$\tilde{A}_{ij} = \frac{{}_{(2)}F_{ij}}{a^3} \int_{t_*}^t a(t')dt' + \frac{{}_{(2)}C_{ij}(x^k)}{a^3} + O(\epsilon^4), \quad (C13)$$

$${}_{(2)}F_{ij} \equiv \frac{1}{{}_{(0)}L^4} \left[{}_{(2)}\bar{R}_{ij} + {}_{(2)}R_{ij}^L - \frac{{}_{(0)}f_{ij}}{3}{}_{(0)}f^{kl}({}_{(2)}\bar{R}_{kl} + {}_{(2)}R_{kl}^L) \right], \quad (C14)$$

$${}_{(2)}R_{ij}^L \equiv -\frac{2}{{}_{(0)}L}\bar{D}_i\bar{D}_j{}_{(0)}L - \frac{2}{{}_{(0)}L}{}_{(0)}f_{ij}\bar{\Delta}{}_{(0)}L + \frac{6}{{}_{(0)}L^2}\bar{D}_i{}_{(0)}L\bar{D}_j{}_{(0)}L - \frac{2}{{}_{(0)}L^2}{}_{(0)}f_{ij}\bar{D}_k{}_{(0)}L\bar{D}^k{}_{(0)}L, \quad (C15)$$

where t_* is an arbitrary reference time, ${}_{(0)}f_{ij}$ is an arbitrary and symmetric tensor of the spatial coordinates, \bar{R}_{ij} is the Ricci tensor with respect to ${}_{(0)}f_{ij}$, \bar{D}_i is the covariant derivative with respect to ${}_{(0)}f_{ij}$, $\bar{\Delta}$ is the Laplacian with respect to ${}_{(0)}f_{ij}$, and ${}_{(2)}C_{ij}$ is an arbitrary, symmetric and traceless tensor which depends only on the spatial coordinates.

2. The solution to $O(\epsilon^2)$ in gradient expansion

Now we consider the general solution valid to $O(\epsilon^2)$ in gradient expansion. As we have seen in the previous subsection, among the basic variables we have introduced, the only quantities whose leading order terms are lower than $O(\epsilon^2)$ are ψ , $\tilde{\gamma}_{ij}$, α , and ϕ . Hence what we have to do is to evaluate the next order terms of these variables.

First let us consider $\tilde{\gamma}_{ij}$. Substituting Eq. (C13) in Eq. (C8), and choosing $\beta^i = O(\epsilon^3)$, we obtain the general solution for $\tilde{\gamma}_{ij}$,

$$\tilde{\gamma}_{ij} = {}_{(0)}f_{ij}(x^k) - 2{}_{(2)}F_{ij}(x^k) \int^t \frac{dt'}{a^3(t')} \int^{t'} a(t'') dt'' - 2{}_{(2)}C_{ij}(x^k) \int^t \frac{dt'}{a^3(t')} + O(\epsilon^4) + {}_{(2)}f_{ij}(x^k), \quad (C16)$$

where ${}_{(2)}f_{ij}$ is an arbitrary and symmetric tensor which depends only on the spatial coordinates. We may absorb it into ${}_{(0)}f_{ij}$ without loss of generality. Thus we have

$$\tilde{\gamma}_{ij} = {}_{(0)}f_{ij}(x^k) - 2{}_{(2)}F_{ij}(x^k) \int^t \frac{dt'}{a^3(t')} \int^{t'} a(t'') dt'' - 2{}_{(2)}C_{ij}(x^k) \int^t \frac{dt'}{a^3(t')} + O(\epsilon^4), \quad (C17)$$

Here, we note that ${}_{(0)}f_{ij}$ is not completely arbitrary, but its determinant must be unity, $\det({}_{(0)}f_{ij}) = 1 + O(\epsilon^4)$.

To obtain the other variables, we first consider the solution for ${}_{(2)}\phi$. We express ${}_{(2)}\alpha$ in terms of ${}_{(2)}\phi$ by using Eq. (C10). We obtain

$${}_{(2)}\alpha = \frac{2}{\dot{\phi}^2} \left(2\dot{\phi} {}_{(2)}\dot{\phi} - \frac{dV}{d\phi} {}_{(2)}\dot{\phi} \right) \quad (C18)$$

Inserting this into the field equation (C4), we obtain a closed equation for ${}_{(2)}\phi$,

$${}_{(2)}\ddot{\phi} - \frac{H\dot{\phi} + 2dV/d\phi}{\dot{\phi}} {}_{(2)}\dot{\phi} - \frac{\dot{\phi} d^2V/d\phi^2 + 2HdV/d\phi}{\dot{\phi}} {}_{(2)}\phi = 0. \quad (C19)$$

Although this equation looks difficult to solve at first glance, it turns out that it can be analytically solved. Here, we note that because the lowest order scalar field is only a function of time, the above equation is exactly the same as the one in the long wavelength limit of linear theory. Then, it is known that there exists a particular solution that satisfies

$$\dot{\phi} {}_{(2)}\dot{\phi}_d - \frac{dV}{d\phi} {}_{(2)}\phi_d = 0, \quad (C20)$$

where the subscript d is attached since it corresponds to a decaying mode solution in the linear limit. This may be integrated easily to give

$${}_{(2)}\phi_d \propto v(t) \equiv \exp \left[\int^t dt \frac{dV/d\phi}{\dot{\phi}} \right] = \frac{\dot{\phi}(t_*)}{a^3(t)\dot{\phi}(t)}, \quad (C21)$$

where we have normalized the amplitude so that $v = 1/a^3(t_*)$ at $t = t_*$.

Once we know a particular solution, the other independent solution, $u(t)$, can be found by the use of the Wronskian. For two independent solutions u and v of Eq. (C19), the Wronskian,

$$W = \dot{u}v - \dot{v}u, \quad (C22)$$

satisfies

$$\dot{W} + b(t)W = 0, \quad (C23)$$

where $b(t)$ is the coefficient of ${}_{(2)}\dot{\phi}$ in Eq. (C19),

$$b \equiv -\frac{H\dot{\phi} + 2dV/d\phi}{\dot{\phi}}. \quad (\text{C24})$$

Thus we obtain

$$\dot{u}v - \dot{v}u = W \propto \exp\left[-\int^t b(t)dt\right] = \frac{a}{(a^3\dot{\phi})^2} \propto av^2. \quad (\text{C25})$$

Then since

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{\dot{u}v - \dot{v}u}{v^2} = \frac{W}{v^2}, \quad (\text{C26})$$

we readily find

$$u = v \int_{t_*}^t \frac{W}{v^2} dt' = \frac{\dot{\phi}(t_*)}{a^3(t)\dot{\phi}(t)} \int_{t_*}^t a(t') dt', \quad (\text{C27})$$

where we have normalized u so that $(u/v) = a(t_*)$ at $t = t_*$. Thus, the general solution for ${}_{(2)}\phi$ is given by

$${}_{(2)}\phi = C(x^k)u(t) + D(x^k)v(t). \quad (\text{C28})$$

Given the general solution for ${}_{(2)}\phi$, the remaining variables ${}_{(2)}\alpha$ and ${}_{(2)}\psi$ are determined as follows. Equation (C18) gives the solution for ${}_{(2)}\alpha$ as

$$\begin{aligned} {}_{(2)}\alpha &= \frac{2}{\dot{\phi}^2} \left[\left(2\dot{\phi}\dot{u} - \frac{dV}{d\phi}u \right) C(x^k) + \left(2\dot{\phi}\dot{v} - \frac{dV}{d\phi}v \right) D(x^k) \right] \\ &= \frac{2\dot{\phi}_*}{\dot{\phi}^3 a^3} \left[\left(2a\dot{\phi} + \frac{dV}{d\phi} \int_{t_*}^t a(t') dt' \right) C(x^k) + \frac{dV}{d\phi} D(x^k) \right]. \end{aligned} \quad (\text{C29})$$

In terms of this solution, Eq. (C7) gives the $O(\epsilon^2)$ part of ψ as

$$\dot{\psi} = \frac{H}{2} {}_{(0)}L {}_{(2)}\alpha, \quad (\text{C30})$$

where we have set $\beta^i = O(\epsilon^3)$. Integrating this equation, we obtain

$$\psi = {}_{(0)}L \left[1 + \frac{1}{2} \int_{t_*}^t {}_{(2)}\alpha H dt' \right] + {}_{(2)}L(x^k), \quad (\text{C31})$$

where ${}_{(2)}L(x^k)$ is an arbitrary spatial function of $O(\epsilon^2)$, which we may absorb into ${}_{(0)}L$ without loss of generality.

Up to now we have not considered the Hamiltonian and momentum constraint equations. The constraint equations will relate the quantities ${}_{(0)}L$, ${}_{(0)}f_{ij}$, ${}_{(2)}C_{ij}$, ${}_{(2)}C$ and ${}_{(2)}D$. First let us focus on the Hamiltonian constraint, Eq. (C5). Using Eqs. (C10) and (C11), it becomes

$$\frac{1}{a^2 {}_{(0)}L^5} [-8 {}_{(0)}f^{ij} \bar{D}_i \bar{D}_j {}_{(0)}L + {}_{(0)}f^{kl} {}_{(2)}\bar{R}_{kl} {}_{(0)}L] = 48\pi G \left[-\dot{\phi} {}_{(2)}\dot{\phi} + \frac{dV}{d\phi} {}_{(2)}\phi \right]. \quad (\text{C32})$$

Comparing this equation with Eq. (C20), we see that the right-hand side vanishes for the decaying mode solution v . Thus inserting the general solution (C28) into the above, we obtain

$$\frac{1}{a^2 {}_{(0)}L^5} [-8 {}_{(0)}f^{ij} \bar{D}_i \bar{D}_j {}_{(0)}L + {}_{(0)}f^{kl} {}_{(2)}\bar{R}_{kl} {}_{(0)}L] = -48\pi G \frac{\dot{\phi}(t_*)}{a^2} C(x^k). \quad (\text{C33})$$

Thus $D(x^k)$ is not constrained by the Hamiltonian, while $C(x^k)$ is determined in terms of ${}_{(0)}L$ and ${}_{(0)}f_{ij}$ as

$$\begin{aligned} {}_{(2)}C &= \frac{-1}{48\pi G {}_{(0)}L^5 \dot{\phi}_*} [-8 {}_{(0)}f^{ij} \bar{D}_i \bar{D}_j {}_{(0)}L + {}_{(0)}f^{kl} {}_{(2)}\bar{R}_{kl} {}_{(0)}L] + O(\epsilon^4) \\ &= \frac{-1}{48\pi G {}_{(0)}L^4 \dot{\phi}_*} {}_{(0)}f^{kl} [{}_{(2)}R_{kl}^L + {}_{(2)}\bar{R}_{kl}] + O(\epsilon^4) \\ &= \frac{-1}{48\pi G \dot{\phi}_*} R[L^4 f] + O(\epsilon^4), \end{aligned} \quad (\text{C34})$$

where $\dot{\phi}_* = \dot{\phi}(t_*)$ and $R[L^4 f]$ is the Ricci scalar of the metric ${}_{(0)}L^4{}_{(0)}f_{ij}$.

The $O(\epsilon^3)$ part of the momentum constraint (C6) yields

$$\bar{D}^j [{}_{(0)}L^6{}_{(2)}F_{ij}] = -8\pi G_{(0)}L^6\dot{\phi}_*\partial^j ({}_{(2)}C(x^k)) , \quad (C35)$$

$$\bar{D}^j [{}_{(0)}L^6{}_{(2)}C_{ij}] = -8\pi G_{(0)}L^6\dot{\phi}_*\partial^j ({}_{(2)}D(x^k)) . \quad (C36)$$

The latter equation implies ${}_{(2)}D$ is not arbitrary but expressed in terms of ${}_{(0)}L$, ${}_{(0)}f_{ij}$ and ${}_{(2)}C_{ij}$, while the former equation is found to be consistent with Eq. (C34). This consistency is a result of the Bianchi identities.

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